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University of Wisconsin-Madison

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Air Force Office of Scientific Research
(7/1/83 through 9/30/84)

Final Report

Submitted in fulfillment of the requirements of AFOSR Grant No. 83-0281

By

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Principal Investigator

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Citing data collected and analyzed with the University of Wisconsin MIPAC (Modelling, Information Processing and Control) Facility electronic measurement and analysis equipment, the investigators make some general comments concerning mathematical models which appear to be appropriate for modelling certain vibratory systems of distributed parameter type. Aspects discussed include: location of vibrational spectra, damping rates, and spectral displacement due to mass density and/or elasticity variations. Particular emphasis is placed on some properties of segmented beams. The article ends with a preliminary mathematical discussion of the feasibility of parameter identification, from vibrational spectrum data alone, in the wave and Euler-Bernoulli beam equations.

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CONTENTS

I. Financial Report.

The University of Wisconsin Modelling, Information Processing and Control (MIPAC) Facility began operation in October, 1983, with initial acquisitions of equipment provided for under the DoD University Instrumentation Program. The equipment was funded by two separate grants, one from the Air Force Office of Scientific Research (Grant No. AFOSR-83-0281, $110,000) and one from the Army Research Office (Grant No. DAAG29-83-G-0056, $26,000). The following items were acquired from Hewlett-Packard, Inc. under Grant No. AFOSR-83-0281 in accordance with University of Wisconsin policies and procedures.

	Cost
1. Purchase Order No. UAD 078Y002 Dated: 9/29/83	$101,706.01

This purchase order covered the initial acquisition of our Hewlett Packard 5451C Fourier Analyzer and related peripheral equipment. Specifically, the items purchased from Hewlett-Packard, Inc. are as follows:

5451C Fourier Analyzer, Options 600 & S23
10833B HPIB Cable
10697D Field Add-on: S83 Var Resis Scope
620 DAC Module
350 Vibration Con
451 Less 5477A

After some months of initial operation it became clear that more flexible use of the Hewlett-Packard 5451C Fourier Analyzer required acquisitions and installation of a different operating system for the attached HP 1000 computer and a tape cassette unit in the Fourier Analyzer's HP 2648A graphics terminal. The complete acquisition from Hewlett-Packard, Inc. which covered these, and related, items is summarized below:

2. Purchase Order No. UAD 924C055 Dated: 6/30/84	6395.50

12747H 128 KByte Memory Board
92068A RTE IVB System
92068A Opt. 031, 7906 Disc Media
12539C Time Base Generator
12966A Buffered Asynchronous Interface, Option 001
13236B Cartridge Tape Refit for HP2648A Graphics Terminal
13261A 2648A ROMS (Software for Cartridge Tape Unit) ITR 252

The final acquisitions under the subject grant were three disk cartridges and four boxes of tape cassetes for use with the HP 5451C's hard disk unit and tape cassette unit, respectively. These are listed below.

3. Purchase Order No. UAE 924C545 Dated: 9/27/84	980.00

5 Cartons HP 98200 A Cartridge Tapes
4 HP 12940 A Disc Cartridges

TOTAL	$109,081.51

II. Scientific Report

MODELLING AND PARAMETER ESTIMATION FOR DISTRIBUTED VIBRATORY SYSTEMS;*

A Report on Research Projects at the MIPAC Facility of the University of Wisconsin, Madison, to June 30, 1984

by D. L. Russell**

ABSTRACT

Citing data collected and analyzed with the University of Wisconsin MIPAC (Modelling, Information Processing and Control) Facility electronic measurement and analysis equipment, we make some general comments concerning mathematical models which appear to be appropriate for modelling certain vibratory systems of distributed parameter type. Aspects discussed include: location of vibrational spectra, damping rates, and spectral displacement due to mass density and/or elasticity variations. Particular emphasis is placed on some properties of segmented beams. The article ends with a preliminary mathematical discussion of the feasibility of parameter identification, from vibrational spectrum data alone, in the wave and Euler-Bernoulli beam equations.

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*Supported through the DoD University Research Instrumentation Program (FY 83): Air Force Office of Scientific Research under Grant No. AFOSR-83-0281 and Army Research Office under Grant No. DAAG29-83-G-0056.

**Mathematics Research Center, University of Wisconsin, Madison, WI 53706, U.S.A.

MODELLING AND PARAMETER ESTIMATION
FOR DISTRIBUTED VIBRATORY SYSTEMS

1. Introduction.

Beginning in October, 1983, we have been developing the University of Wisconsin Modelling, Information Processing and Control (MIPAC) Facility with the cooperation of the Department of Mathematics, the Department of Electrical and Computer Engineering and the Mathematics Research Center. Our objectives in the operation of this Facility are as follows:

1. Better understanding of physical processes:
   - (a) From the dynamical point of view;
   - (b) From the point of view of measurement capabilities and limitations;
   - (c) From the point of view of our ability to model the process mathematically.
2. Development of modelling techniques:
   - (a) Qualitative model identification;
   - (b) Model calibration and parameter estimation.
3. Development of information processing techniques:
   - (a) Instrument capabilities and limitations;
   - (b) Noise limitations;
   - (c) Discretization problems;
   - (d) Mathematical information processing.
4. Development of real time control techniques:
   - (a) Actuator capabilities and limitations;
   - (b) Real time computational capabilities and limitations, including discretization problems;
   - (c) Mathematical control law development.

Initial grants from the Air Force Office of Scientific Research and the Army

Research Office, with additional assistance from the Wisconsin Alumni Research Foundation, have enabled us to acquire and begin operation of a two unit Facility, incorporating an Analysis Unit and a Model Development Unit. The electronic equipment acquired includes a Hewlett-Packard 5451C Fourier Analyzer, a Bruel & Kjaer Vibration Exciter and supporting peripheral units. The University is providing us with approximately 1500 square feet of newly remodelled laboratory space which we expect to overcrowd very quickly. At present two faculty members and two research assistants are working with the facility and we expect to add an additional senior staff member in the autumn of 1984.

The paragraphs and figures which follow do not constitute a research paper as such but, rather, a description of the work which we have been carrying out in the MIPAC Facility, indications for future modelling work, both of a mathematical character and of the laboratory sort, and sketches of some of the mathematical questions likely to enter into our work. Not everything is reported here; we have not, for example, discussed our extensive laboratory work with various types of nonlinear oscillators.

One point which we wish to make very strongly is the way in which physical data can prompt and direct mathematical investigation in unexpectedly significant ways. We will illustrate this in Section 3 by exhibiting data obtained from segmented beams and discussing, briefly, the implications as regards the mathematical work required to provide an anlytical framework to correspond to and explain what we observe.

In Section 2 of this article we discuss data obtained from vibrations of uniform beams. This provides a necessary background for Section 3 and also allows us to discuss important laboratory evidence in regard to the relationship between the frequency of a mode of vibration and the rate at which that mode of vibration is attenuated through internal energy decay.

The last section of this article, Section 4, is an analysis of the feasibility of first order mass density estimation/identification on the sole basis of observed spectral data. We will see how a very practical engineering problem leads to some quite interesting and intricate mathematical questions connected with completeness and linear independence of certain sets of functions derived from the eigenfunctions of a vibrating system.

We will freely use terminology common in the literature of discrete Fourier methods. The reader is referred to [5] for definition of these terms and development of the related theory.

2. Frequency Distribution and Damping for Uniform Thin Beams.

There are a number of ways in which the natural frequencies of vibration of an elastic body may be determined. The one which we, along with others, have used in connection with various types of beams is extremely simple but quite effective. The beam is tapped, once, and quite smartly, with an appropriate hammer or similar instrument. The resulting vibrations of the beam are transformed into electric signals by an accelerometer attached to the beam. The data acquisition functions of the Fourier analyzer are triggered and the accelerometer signals are read in and recorded. A record such as is shown in Figure 1 is obtained. The corresponding discrete Fourier transform and the associated power spectrum are computed. This procedure is repeated a number of times, typically 25 to 50, and an averaged power spectrum is obtained, whose logarithm is then computed and displayed, as in Figure 2. In the process the beam is struck at random points along its length to ensure excitation of all modes.

The log averaged power spectrum (LAPS) for a thin beam (approximately 2 mm. thick, 25 mm. wide and 1 meter long) of mild steel, clamped at one end and free at the other, obtained as indicated above, is shown in Figure 2 for the frequency range 0 - 2500 Hz. Depending on where one stops counting, 11 or 12 distinct modes of vibration, whose associated natural frequencies are indicated by the sharp peaks in the LAPS, appear. Beyond about 1200 Hz. the peaks become indistinguishable from the ambient noise and are no longer significant. The measurement used in this case was the acceleration of the beam at the free boundary.

In Figure 3 we show the same spectrum, now restricted to the range 0 - 1000 Hz., in order to compare it with similar LAPS obtained from other types of beams. Figure 4 exhibits the spectrum from a thin aluminum beam. This beam is highly flexible and, as a result, the number of spectral peaks, corresponding to natural

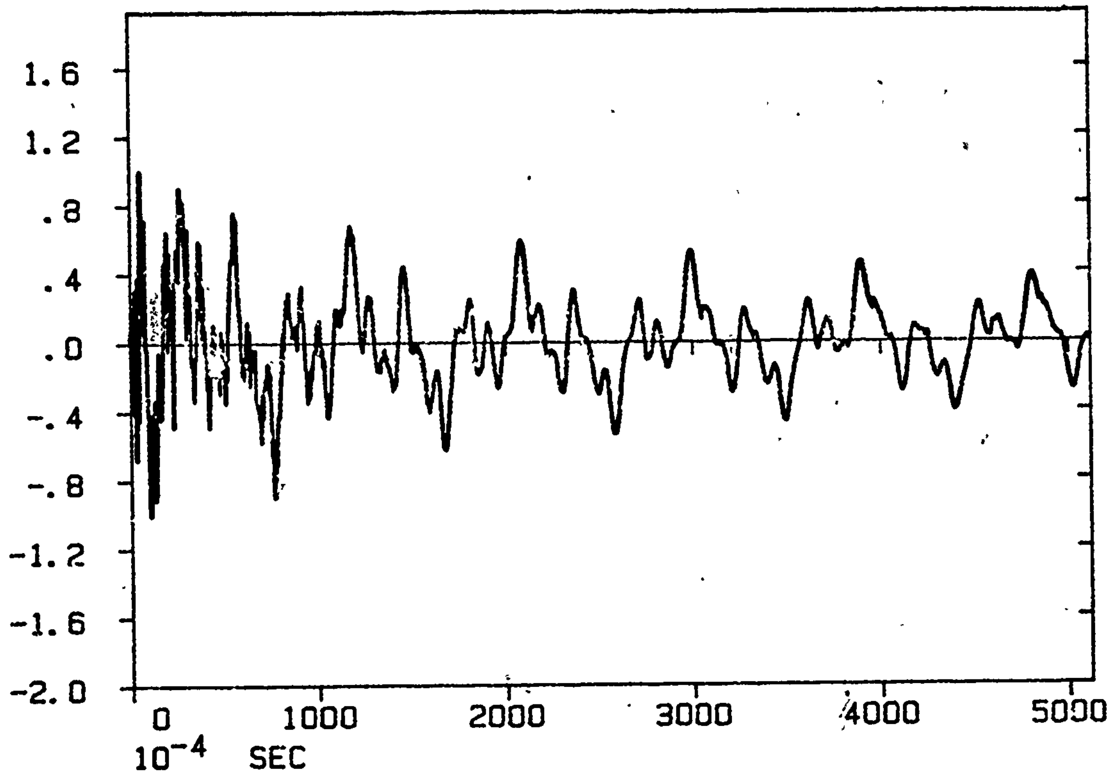


Figure 1

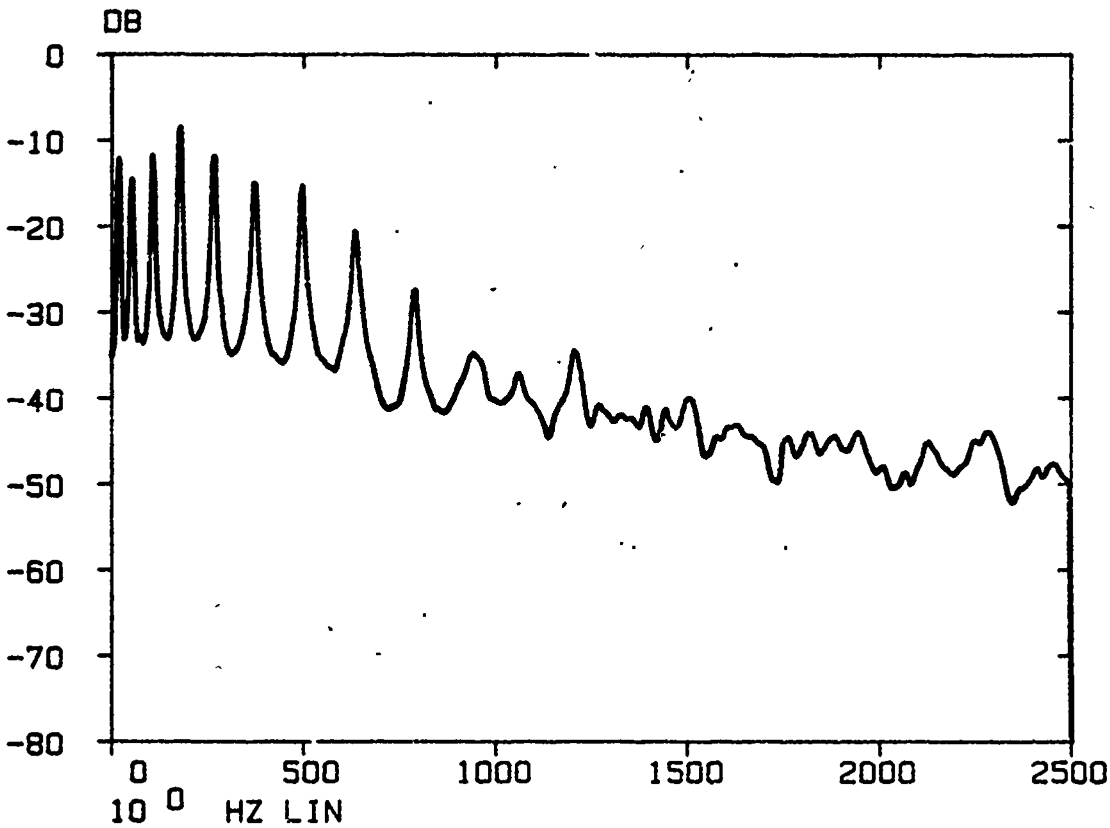


Figure 2

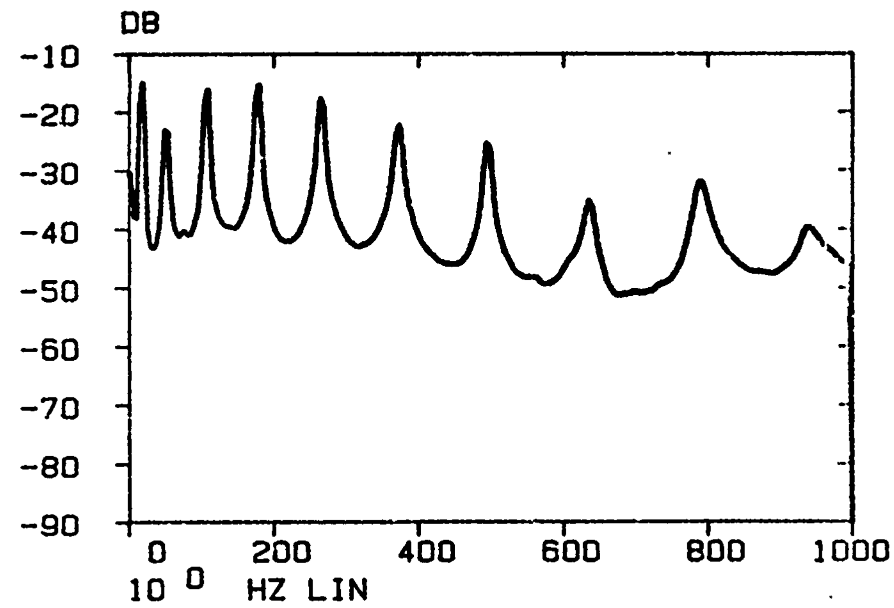
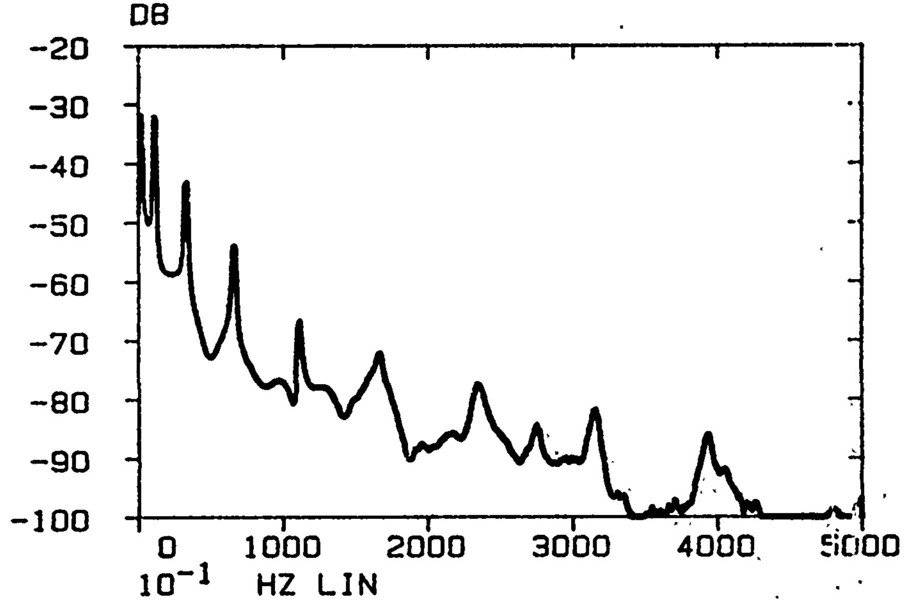


Figure 3



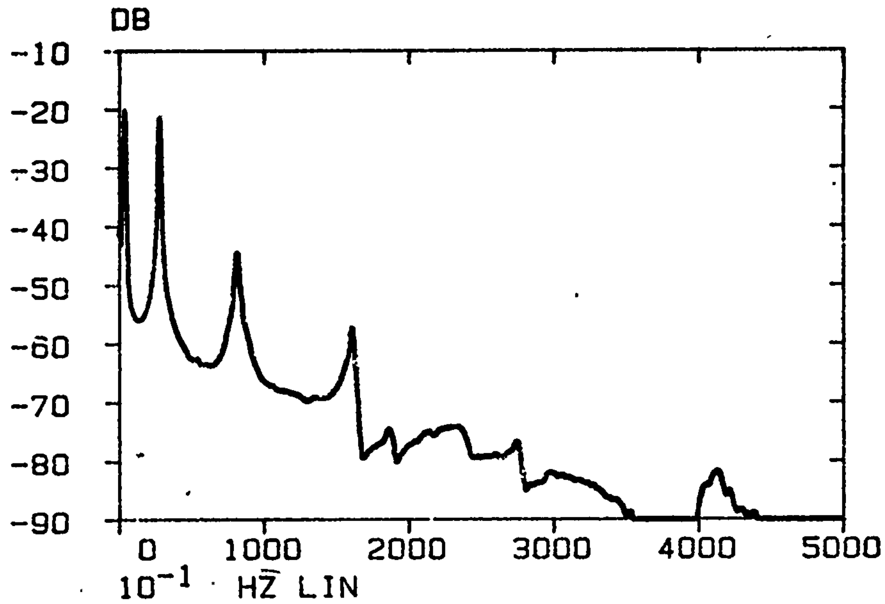
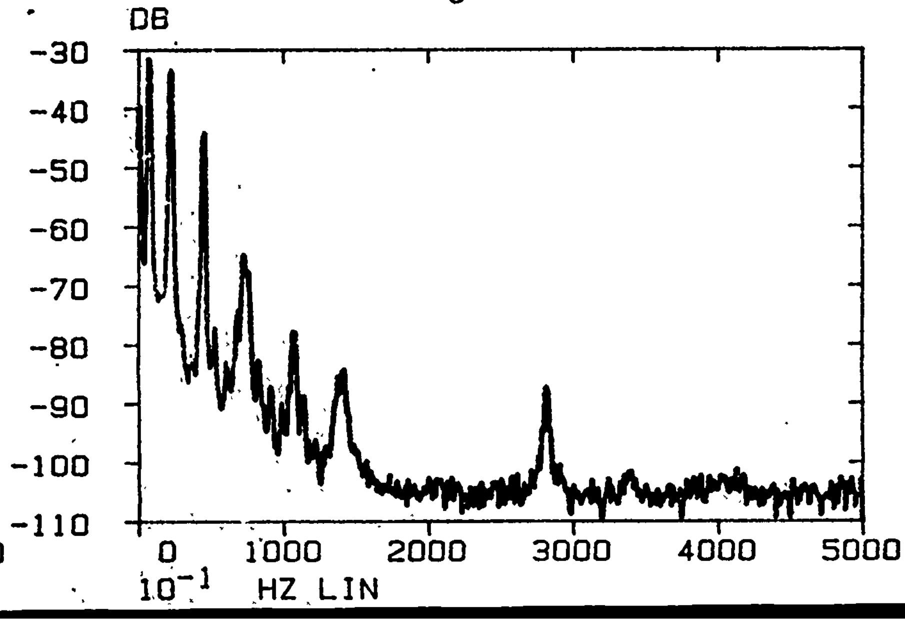


Figure 5



frequencies, occurring in the range 0 - 500 Hz. here is comparable to the number occurring in twice that range in Figure 3. In Figure 5 we see the LAPS obtained from a thin hardwood beam. In this case a much smaller number of natural modes of vibration is observable, perhaps because higher modes are over-damped in this case. In Figure 6 we see the spectrum obtained from a segmented wooden beam, actually a good quality carpenter's rule obtained at a local hardware store. The most significant feature occurring in this case, the gap between approximately 140 Hz. and 280 Hz., will be discussed in Section 3 below.

The decay of amplitude with increasing frequency evident in Figures 4 through 6, in contrast to Figure 3, is not particularly significant. Figure 3 was obtained using acceleration data while Figures 4 through 6 were obtained from velocity data.

Since we are, eventually, interested in control applications, we will place particular emphasis on the properties of systems which are important in this connection. A very old argument, re-disputed by every new generation of control theorists, concerns the necessity and/or advisability of distributed parameter representation of spatially extended vibrating continua such as beams, plates, shells, etc. We are hardly prepared to pronounce judgment on this unresolvable question but we are in a position to make a few relevant remarks.

First of all, it is evident that any reasonable answer to the question must take into account the mode of anticipated excitation and the mode of output evaluation. Here we restrict attention to the latter. Figure 3 corresponds to acceleration measurements. Here the higher frequencies do not fall off very markedly, in contrast to Figures 4 through 6, taken with velocity measurements where, if we take into account that this is a logarithmic plot, nothing beyond the first two or three modes is of much significance.

If we assume the worst case (best case for the proponents of distributed parameter theory) corresponding to Figure 3, we see that adequate modal description of the system might require use of a linear system whose dimension is on the order of 20. A distributed parameter approach necessarily presupposes the use of some approximation procedure, e.g., one based on spline functions, in the final application. Assuming approximation via cubic splines, a system dimension of 20 allows subdivision of the spatial domain into about 8 subintervals. Comparison should be made between modelling and control results obtained with this number of subintervals and corresponding results obtained using a modal representation. The frequencies present in the model will not be identical in the two approaches and, perhaps more importantly, the control and/or disturbance input coefficients ([6]) obtained via the two approaches will differ. We feel that it will be interesting to see how the performances compare when the two contenders are matched to each other in this way.

Without pre-judging the outcome of the contest we have just proposed, let us proceed under the assumption that a distributed parameter model is to be used. The simplest thin beam model based on partial differential equations is provided by the Euler-Bernoulli equation

$$\rho w_{tt} + EI\, w_{xxxx} = 0,$$

where ρ is the mass per unit length and EI is the bending modulus. In the unforced, clamped case, which corresponds to our data, the appropriate boundary conditions are

$$w(0,t) = 0, \quad w_x(0,t) = 0,$$

$$w_{xx}(L,t) = 0, \quad w_{xxx}(L,t) = 0,$$

where L is the length of the beam. With

$$w(x,t) = e^{i\omega t} z(x)$$

we find that

$$-\rho\omega^2 z + EIz_{xxxx} = 0$$

and with $\alpha = (\rho/EI)^{1/4}$ we must have

$$z(x) = c_1 \sin(\alpha\omega^{1/2} x)+c_2 \cos(\alpha\omega^{1/2} x)+c_3 \sinh(\alpha\omega^{1/2} x)+c_4 \cosh(\alpha\omega^{1/2} x).$$

The boundary conditions at $x = 0$ imply

$$c_2 + c_4 = 0, \quad c_1 + c_3 = 0,$$

so that, with $c_1 = -c_3 = c$, $c_2 = -c_4 = d$,

$$z(x) = c(\sin(\alpha\omega^{1/2} x) - \sinh(\alpha\omega^{1/2} x)) + d(\cos(\alpha\omega^{1/2} x) - \cosh(\alpha\omega^{1/2} x)).$$

Taking the second and third derivatives at $x = L$ we obtain the equations

$$c(-\sin(\alpha\omega^{1/2} L)-\sinh(\alpha\omega^{1/2} L))+d(-\cos(\alpha\omega^{1/2} L)-\cosh(\alpha\omega^{1/2} L)) = 0,$$

$$c(-\cos(\alpha\omega^{1/2} L)-\cosh(\alpha\omega^{1/2} L))+d(\sin(\alpha\omega^{1/2} L)-\sinh(\alpha\omega^{1/2} L)) = 0.$$

The obvious determinant condition then gives

$$(-\cos(\alpha\omega^{1/2} L)-\cosh(\alpha\omega^{1/2} L))(-\cos(\alpha\omega^{1/2} L)-\cosh(\alpha\omega^{1/2} L))$$

$$-(\sin(\alpha\omega^{1/2} L)+\sinh(\alpha\omega^{1/2} L))(-\sin(\alpha\omega^{1/2} L)+\sinh(\alpha\omega^{1/2} L)) = 0$$

or, taking familiar identities into account,

$$1 + \cos(\alpha\omega^{1/2} L)\cosh(\alpha\omega^{1/2} L) = 0.$$

Examination of the data from which Figure 2 was obtained reveals the following approximate values for the frequencies ω_k and their square roots $(\omega_k)^{1/2}$:

Table 1

ω_k	$(\omega_k)^{1/2}$
2.97	1.72
19.54	4.42
53.72	7.33
107.50	10.37
178.	13.34
266.	16.31
374.	19.34
495.5	22.26
635.	25.20
788.5	28.08
942.	30.69

Since, asymptotically, $\alpha(\omega_k)^{1/2} L \to (k - \frac{1}{2})\pi$, we obtain the approximate value which must be assigned to αL (from averaging the last five values in the table)

$$\alpha L = (1/5)\left(\frac{10.5}{30.69} + \frac{9.5}{28.08} + \frac{8.5}{25.20} + \frac{7.5}{22.26} + \frac{6.5}{19.34}\right)\pi$$
$$= 1.062.$$

using this value to predict ω_1 via solution of the equation provided by the determinant condition we obtain the value

$$\omega_1 = 1.76$$

which may be compared with the observed value 1.72. As the observed value may be significantly lowered by movement of the supports and the mass of the accelerometer, we may regard this as fairly good agreement. Thus the Euler-Bernoulli model appears to be reasonably accurate in predicting the observed frequencies of the beam in this case.

This example does point out, however, one aspect of the distributed parameter approach which must be taken into account. This approach presupposes a certain model or class of models. If exact spectral matching is important it may be a serious disadvantage that no member of the model class is actually able to achieve the observed spectral pattern. This is true for the observed beam frequencies shown above if the Euler-Bernoulli model is assumed at the outset: no spatially constant ρ, EI can achieve exactly the spectral pattern shown, even though the beam is, to every outward appearance, completely uniform. We will have more to say about variable coefficients in Section 4 of this article.

It seems likely that a distributed parameter model, such as the Euler-Bernoulli model or some finite element approximation thereof, will have significant advantages from the point of view of on-line identification because of the smaller number of parameters necessary in order to describe the model, as compared, for example, with a set of equations

$$\ddot{x}_k = -\lambda_k x_k + b_k u \quad (u = \text{the applied control})$$

derived from a modal description. While the latter, because of the large number of parameters involved, permits more or less exact frequency matching, the very multiplicity of parameters present makes for unstable, ill-conditioned identification. The more modest number of parameters involved in a model with an *a priori* assumed structure makes the identification task a much more tractable problem.

The Euler-Bernoulli equation in its commonly written form

$$\rho w_{tt} + EIw_{xxxx} = 0$$

corresponds to conservation of the energy form

$$\mathcal{E} = \frac{1}{2}\int_0^L (\rho(w_t)^2 + EI(w_{xx})^2)dx,$$

where w is lateral deflection and L is the length of the beam. The constants ρ and EI are mass per unit length and bending modulus, respectively. It is well known that energy losses in fact occur and that these losses are frequency dependent. This is amply documented by inspection of Figure 1 wherein it is clear that the high frequency components of the motion are attentuated much more rapidly than are the low frequency components.

A quantitative appreciation of the dependence of damping rate on frequency can be obtained by taking a number of consecutive time records after each excitation of the beam and computing the corresponding power spectra separately. The LAPS may then be overlayed and the vertical positions of the peaks at the same frequency compared in order to estimate the damping rate. In Figure 7, which shows the frequency range 0 - 500 Hz. for the same beam as was used in Figures 2 and 3, four such consecutive spectra are overlayed, the peaks being darkened for easy comparison. The vertical gap between successive peaks is proportional to the damping coefficient for the mode with the indicated frequency. Our experiments indicate a rough linear proportionality between the damping rate and the frequency in modes 1,2,4 and 5, becoming more nearly constant in modes 5 through 7. The anomalous behavior of mode 3 is thought to be due to energy transfer to a torsional vibration mode whose frequency is close to that of the third lateral mode shown. The linear dependence of damping rate on frequency is more impressive in Figure 8, which shows two successive overlayed spectra for a different beam.

A linear damping / frequency ratio corresponds to what is called structural damping in the engineering literature [4]. As shown in [3], it is consistent with the model

$$\rho w_{tt} + \gamma A^{\frac{1}{2}} w_t + Aw = 0$$

where A is the operator EIw_{xxxx} defined on an appropriate domain and $A^{\frac{1}{2}}$ is the unique positive definite square root of A. It is also consistent with computed exponential solutions of the partial differential equation

$$\rho w_{tt} - \gamma w_{txx} + EIw_{xxxx} = 0$$

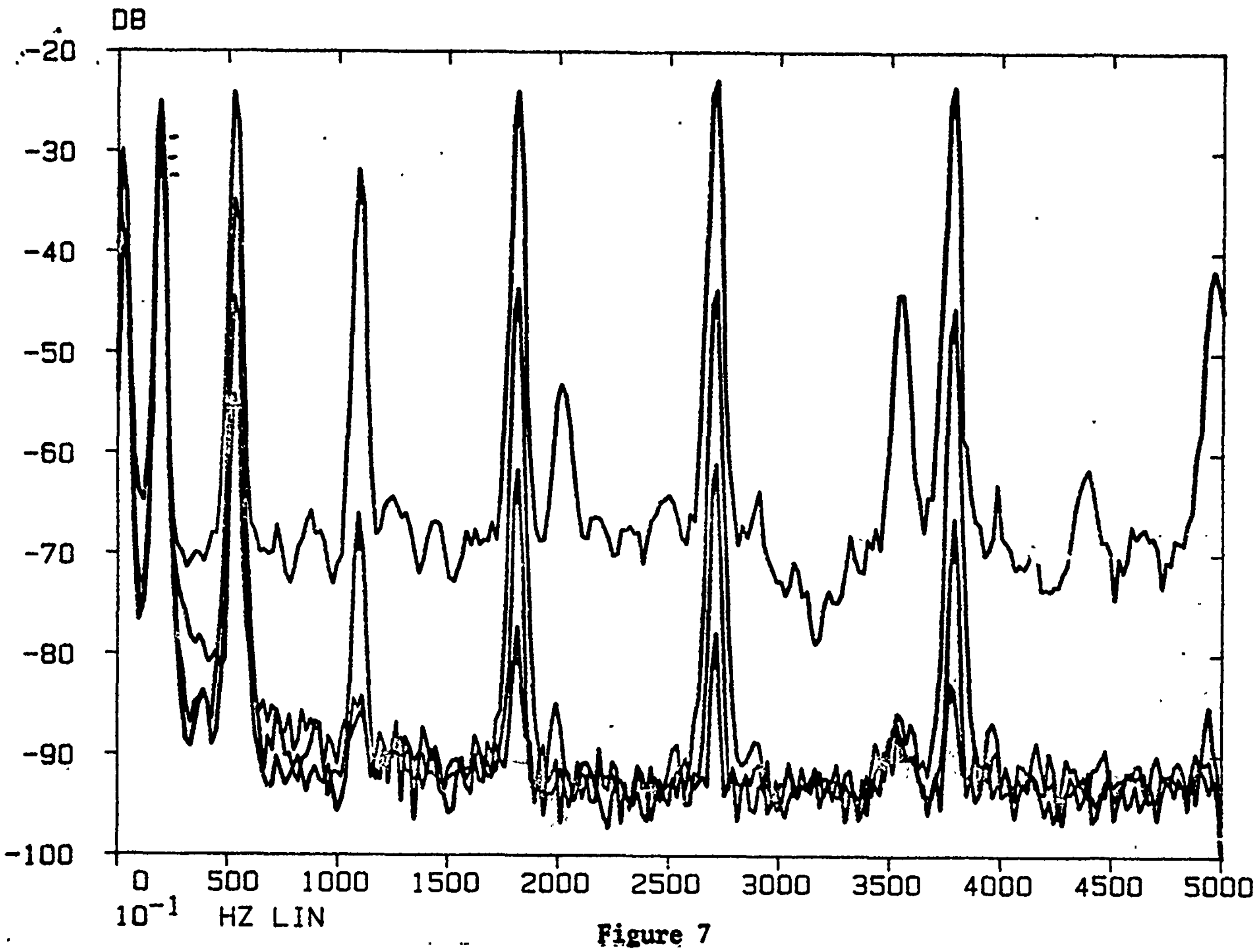


Figure 7

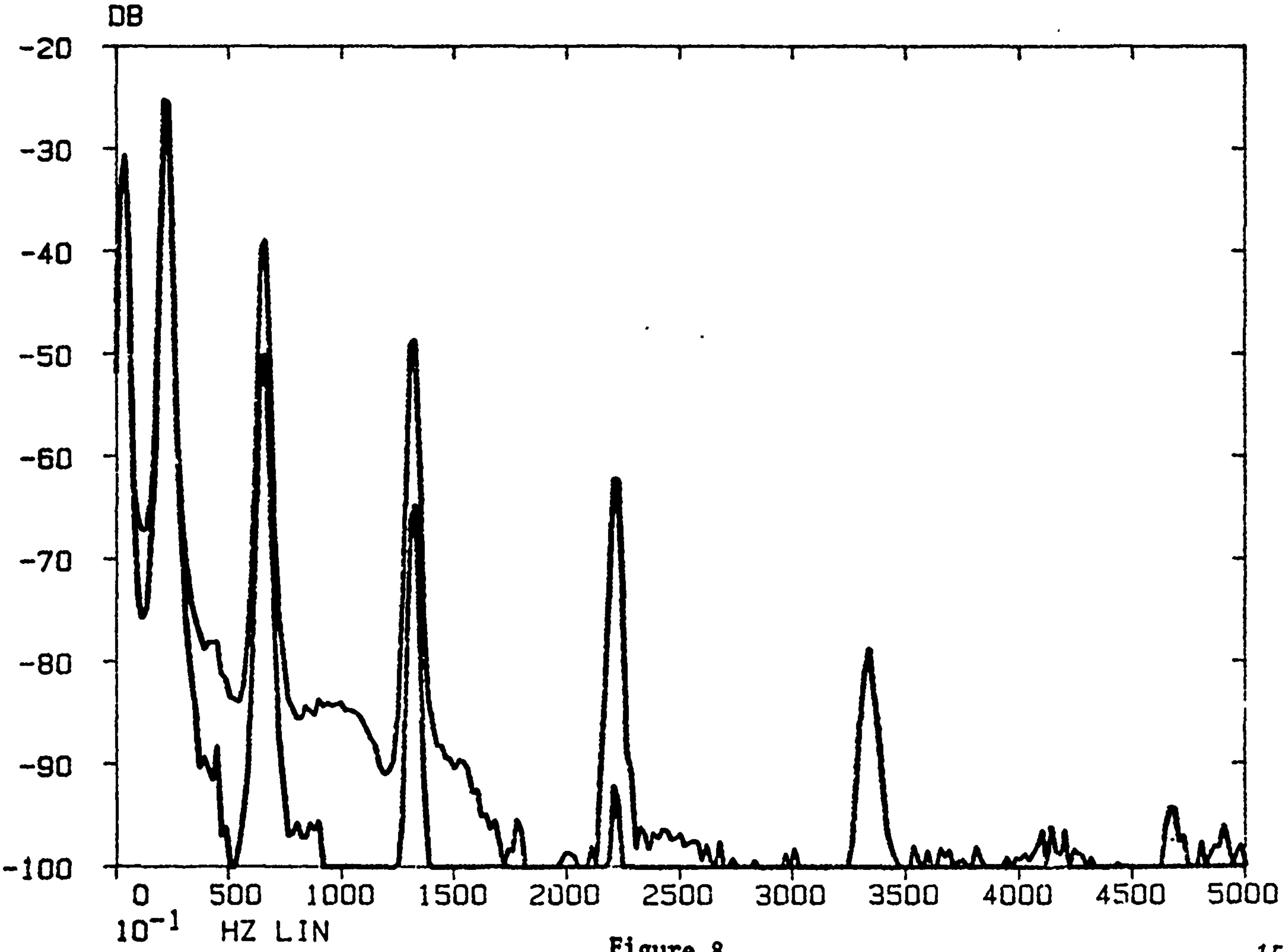


Figure 8

with the boundary conditions

$$w(o,t) = 0, \quad w_x(0,t) = 0,$$

$$EIw_{xxx}(L,t) - \gamma w_{tx}(L,t) = 0, \quad w_{xx}(L,t) = 0.$$

The semigroup theory related to this equation may be based on the energy dissipation law

$$(d/dt)\,\mathcal{E}(w(\cdot,t), w_t(\cdot,t)) = -\gamma\int_0^L (w_{tx}(x,t))^2\,dx \leq 0$$

but it remains to be shown that the generator is a spectral operator, which is necessary if we are to be able to draw conclusions from the spectrum of the operator, and that the semigroup is a holomorphic semigroup. It is clear that this system deserves substantially more attention in the future.

3. Vibration Spectra of Segmented Beams.

Most large structures, including the rather exotic examples that have come to be known as "large space structures", are not single units but are, rather, assemblages of many small units joined together in various ways. Thus, in many respects, simple systems such as the wave equation, Euler-Bernoulli beam equation, thin plate quation, etc , have unrealistically simple structure as compared with real systems. On the other hand it is clear that one cannot develop a theory of distributed parameter systems on the basis of structures of arbitrary complexity; there has to be some unifying pattern and, inevitably, this involves a degree of simplification. One is led to seek out, as a consequence, a class of systems simple enough to admit some hope of analytical treatment and yet possessing enough intricacy to carry one beyond the very simplest models. It seems to us that the models representing the vibration of segmented beams are ideally situated on this middle ground between simplicity and complexity.

For the purposes of this article we will consider a beam occupying a spatial interval which, without loss of generality, we may take to be $[0,\pi]$, and simply supported at $x = 0$ and $x = \pi$. By "simply supported" we mean that the boundary conditions

$$w(0,t) = 0,\ w_{xx}(0,t) = 0,\ w(\pi,t) = 0,\ w_{xx}(\pi,t) = 0$$

apply.

We will suppose that the interval $[0,\pi]$ is divided into N segments of equal length by means of the points

$$x_o = 0,\ x_k = (k\pi)/N,\ k = 1,2,\dots,N.$$

In each of the intervals $I_k = [x_{k-1}, x_k]$, $k = 1,2,\dots,N$, we assume that the Euler-Bernoulli beam equation (here simplified with unit coefficients)

$$w_{tt} + w_{xxxx} = 0$$

is satisfied (whether in a classical or distribution sense need not concern us here) while, at the junction points, x_k, we impose conditions which reflect our assumptions about the nature of the joints there. Here we will distinguish only two different types, shown schematically in Figure 13:

<u>Flexible joints</u>: in this case the energy expression is

$$\mathcal{E} = \tfrac{1}{2}\int_0^{\pi} ((w_t(x,t))^2 + (w_{xx}(x,t))^2)dx + \tfrac{1}{2}\sum_{k=1}^{N} \sigma (w_x(x_k+,t) - w_x(x_k-,t))^2 ,$$

σ denoting the stiffness coefficient for the joints. Conservation of energy requirements then lead to the partial differential equation indicated above and the junction conditions

$$w_{xx}(x_k+,t) = w_{xx}(x_k-,t) = \sigma(w_x(x_k+,t) - w_x(x_k-,t)),$$

$$w_{xxx}(x_k+,t) = w_{xxx}(x_k-,t),$$

$$w(x_k+,t) = w(x_k-,t),\ k = 1,2,\ldots,N-1.$$

Here $f(x+)$ means $\lim_{y\downarrow 0} f(x+y)$, etc.

<u>Massive stiff joints</u>: here we suppose that no macroscopic degree of bending occurs at the joints, which are massively reinforced by structures having very little spatial extent but appreciable mass. In this case, if the reinforcing masses have mass m, the energy expression becomes

$$\mathcal{E} = \tfrac{1}{2}\int_0^{\pi} ((w_t(x,t))^2 + (w_{xx}(x,t))^2)dx + \tfrac{1}{2}m\sum_{k=1}^{N} (w_t(x_k,t))^2 .$$

Energy conservation assumptions then lead to the same partial differential equations as before and the junction conditions

$$w(x_k+,t) = w(x_k-,t),$$

$$w_x(x_k+,t) = w_x(x_k-,t), \quad w_{xx}(x_k+,t) = w_{xx}(x_k-,t),$$

$$mw_{tt}(x_k,t) = w_{xxx}(x_k-,t) - w_{xxx}(x_k+,t), \; k = 1,2,\ldots,N-1.$$

Before discussing the theoretical aspects of these systems, let us consider some of the experimental results. These were obtained with a segmented wooden beam, consisting of short, thin hardwood segments, fastened to each other, end to end (a folding carpenter's rule was used, in fact). The resulting segmented beam was clamped at one end and left free at the other; thus, as far as the boundary conditions are concerned, the experimental apparatus does not exactly match the theoretical model described above. However, we expect the effect of different boundary conditions to be small and quantitative rather than qualitative. The experimental beam has slightly flexible joints, including the joint at the point where it is clamped.

In different experiments, varying numbers of segments of the beam were extended out from the clamping apparatus. In each case the beam was excited by striking and the LAPS was obtained using the Fourier analyzer. The resulting power spectra for beams composed of 5, 7 and 8 projecting segments are shown in Figures 9 through 11, respectively. It will be observed in each case that 5, 7 and 8 vibrational frequencies appear at the low end of the spectrum and then there is a gap until the next frequency appears, as indicated by the peak just to the left of the 300 Hz. point. Another group of frequencies occurs, not too distinctly, from 300 Hz. to 400 Hz., approximately.

In Figure 12 the preceding three figures are superimposed. What is striking is the fact that the first group of 5,7 and 8 frequencies, depending on the number of beam segments, fills approximately the same frequency range in each case,

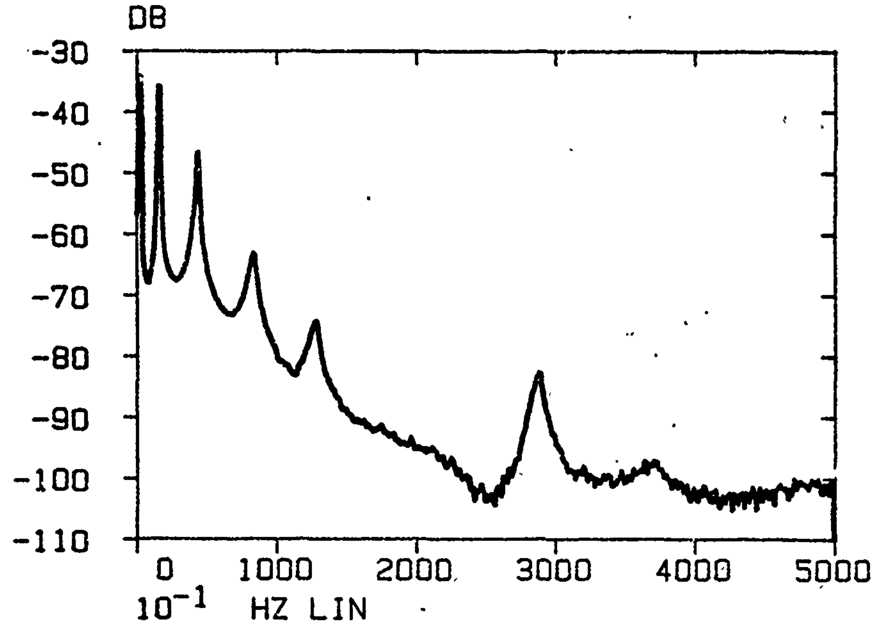


Figure 9

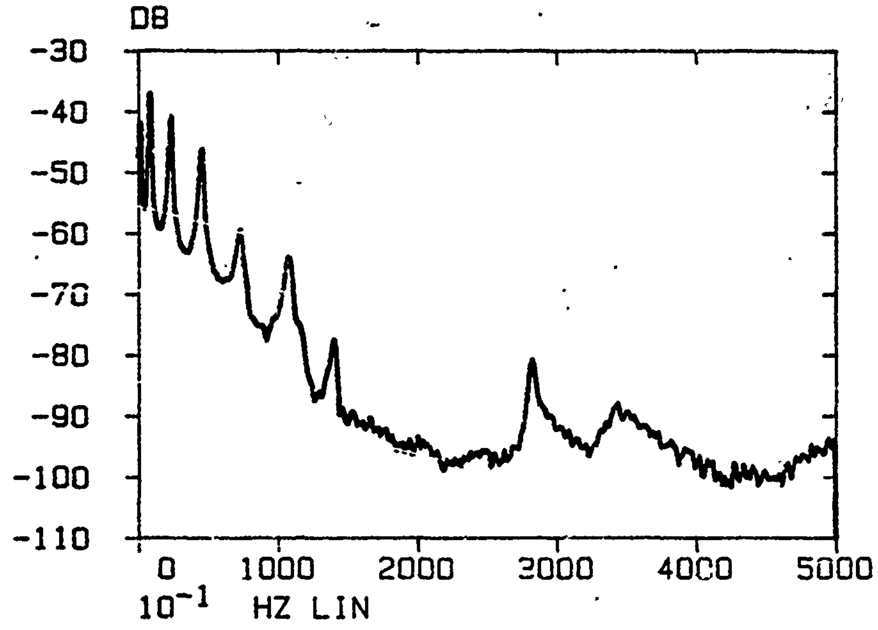


Figure 10

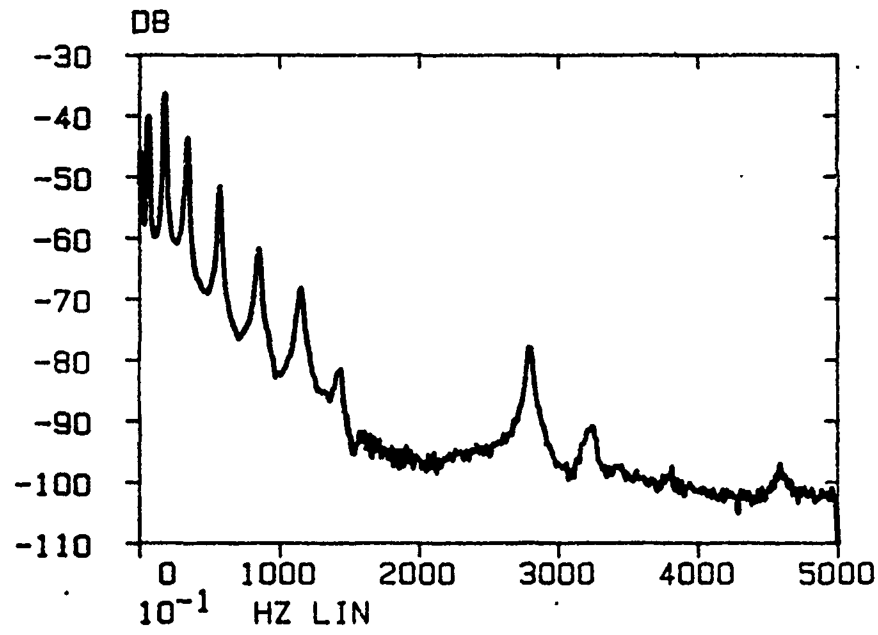


Figure 11

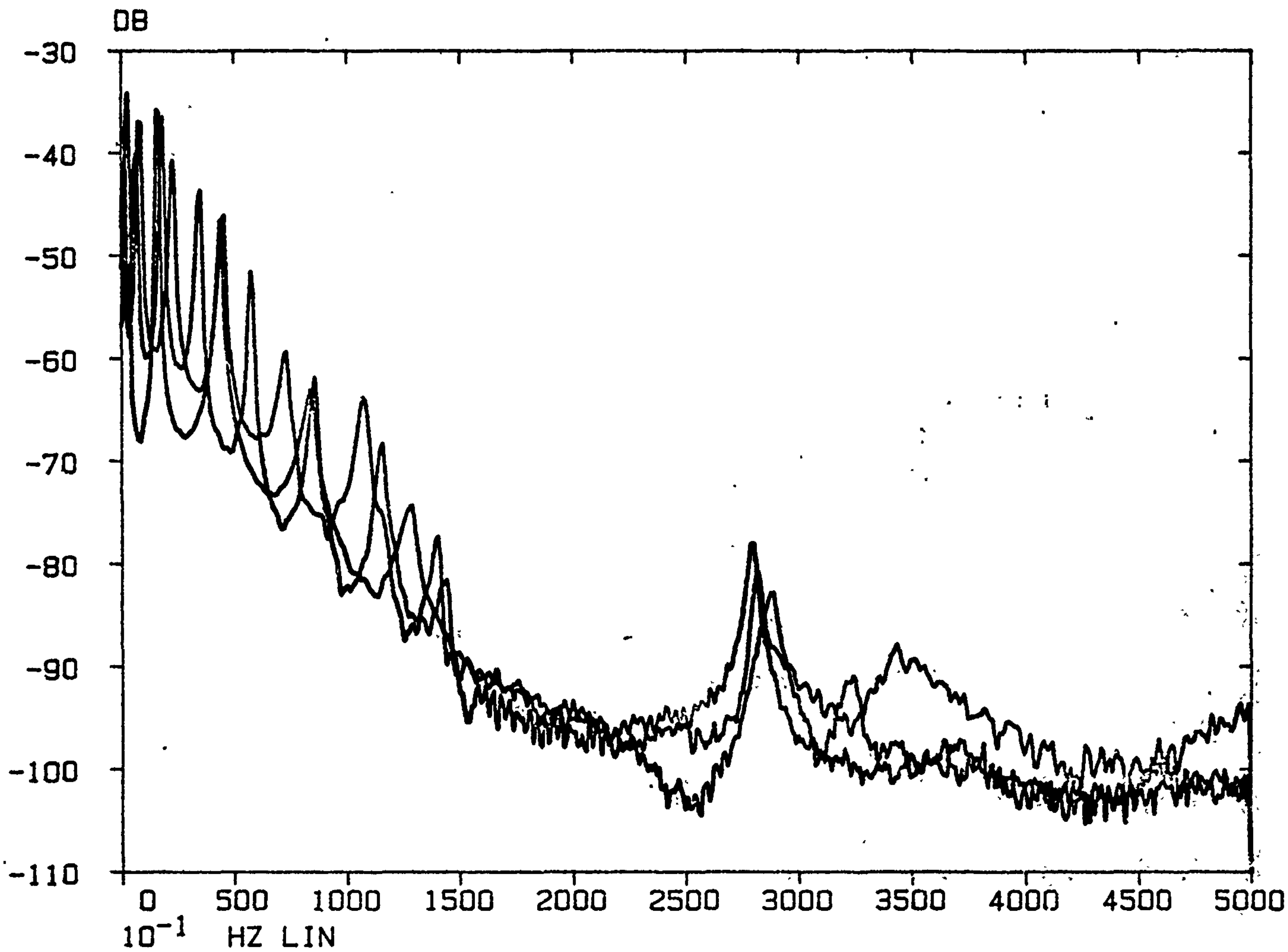


Figure 12

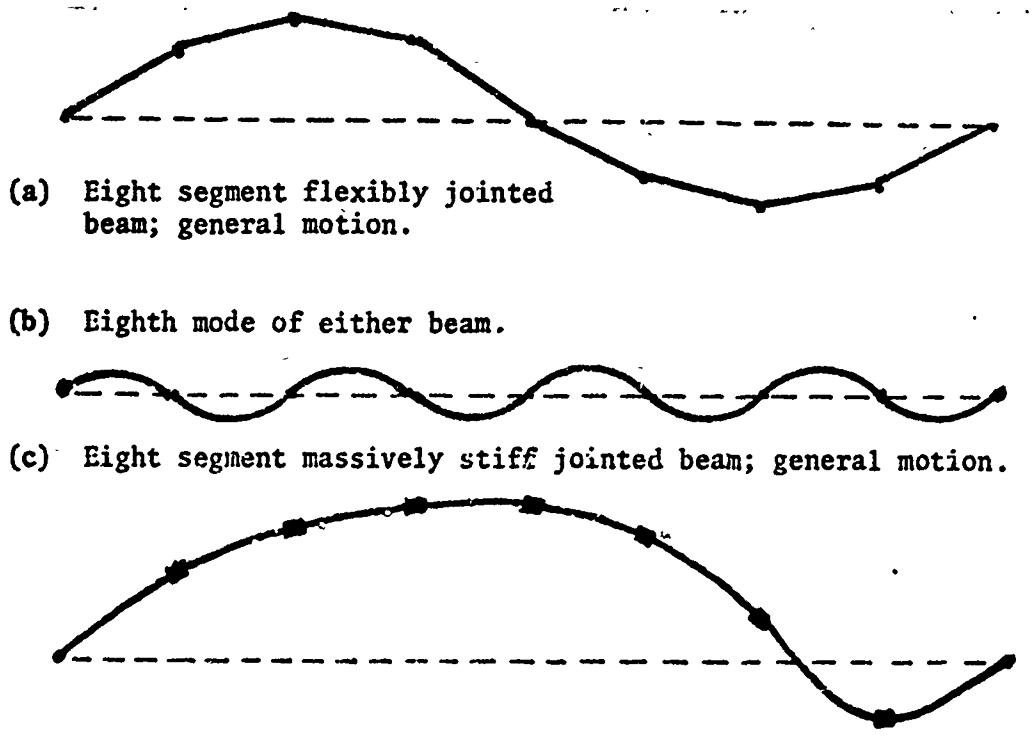


Figure 13

while the next frequency appears at roughly the same location in each case. How is this pattern to be accounted for? We will see that examination of the theorectical model provides an explanation. We will do this only in the context of the flexible joint model but we will make some remarks relative to the massive stiff jointed model as well.

Let us note that in the simply supported, N segment case, if the segments were infinitely stiff by comparison with the joints, we would expect to find a certain number of modes of vibration corresponding to bending of the structure at the joints, the segments remaining straight. It is easy to see that the resulting linear dynamical system has dimension 2(N-1); there will exist N-1 oscillatory modes corresponding to bending at the joints. As the stiffness of the beam segments decreases relative to that of the joints, these vibrational frequencies will decrease and the mode shapes will change somewhat but the character of these first N-1 modes will remain much the same as long as the beam segments remain quite stiff relative to the joints.

Now let us consider the N-th mode. In the unsegmented, uniform case it is easy to see that the modal function, i.e., eigenfunction, in this case is

$$\phi_N(x) = \sin(Nx).$$

However, because this function has zero values and second derivatives at the points x_k, k = 1,2,...,N-1, all of the boundary conditions there imposed on the segmented beam are satisfied by $\phi_N(x)$ and we conclude that this is the N-th eigenfunction in the segmented case as well - no matter what the value of N may be. This reasoning will apply to the 2N-th, 3N-th, etc., modes as well. One can also see that the frequency of vibration of the mN-th mode agrees with the m-th modal frequency of a beam of length π/N. If we pose the same problem on an interval of length $N\pi$, which corresponds to the experimental reality where the length of the beam depends on the number of projecting segments, we then see that the frequency of vibration of the mN-th mode is invariant - it always agrees with that of the m-th mode of vibration of a simply supported beam of length π. This observation corresponds to the near invariance of the (N+1)-st mode in our experiments (the index shifts from N to N+1, however, because the experimental beam is free at one end and a flexible joint exists at the clamped end, thus introducing an additional mode of vibration for the infinitely stiff case).

In the uniform, unsegmented case, the first N-1 frequencies of the simply supported beam would be distributed throughout the interval between 0 and the frequency of the N-th mode, the k-th modal frequency being roughly proportional to k^2. Because of the additional flexibility permitted by bending at the joints, however, those frequencies are all significantly lowered in the segmented case (recall that the k-th eigenvalue of the operator EIw_{xxxx} minimizes the so-called Rayleigh quotient

$$\int_0^{\pi} (w_{xx})^2 dx \Big/ \int_0^{\pi} w^2 dx$$

subject to the imposed boundary conditions and orthogonality to the first k-1 eigenfunctions; additional flexibility results in a numerator which is somewhat smaller relative to the denominator. Since the N-th modal frequency is unchanged as we pass from the uniform to the segmented case, a "gap" opens up between the N-th mode and the (N-1)-st mode. (This corresponds to the gap between the N-th and (N+1)-st modes in the experimental situation which we have described.) Similar gaps may be expected to occur between the mN-th and (mN-1)-st frequencies with the (m-1)N-th through (mN-1)-st frequencies being, relatively speaking, clustered together.

Examination of the second type of system, involving massive, stiff joints, may be expected to uncover a similar pattern of clusters and gaps; again the mN-th modes will not change from the uniform case, the massive joints remaining motionless at the nodes, while the intermediate modes will exhibit depressed frequencies due to the additional inertial effects of those masses (which make the denominator of the Rayleigh quotient larger).

The results sketched here may be expected to have some importance in control applications. The spectral gaps which we have described can be used to advantage to permit control and/or observation of the clustered modes in the first group without excessive interference from the modes in the second, or higher, groups which will be separated from the first group by the spectral gap whose origins we have discussed here. It is clear that the subject deserves more careful and precise mathematical treatment, incorporating estimates and asymptotic expressions for the eigenvalues and eigenvectors of these segmented beam systems. In this connection we are happy to note that some contributions in this direction have recently been made by Chen [2].

4. Parameter Estimates from Frequency Data.

The subject matter of this section is best illustrated by reference to Figure 14. The thin steel beam from which the log-averaged power spectra of Figures 2 and 3 were obtained was first excited in its original configuration and then, after its mass had been modified by the addition of several small permanent magnets near its center, as illustrated in Figure 15. The two power spectra were superimposed to produce Figure 14. In this figure the superimposed power spectra produce double peaks, i.e., peaks lying near each other. In each case the peak to the left corresponds to the beam with additional mass, as would be expected. A very natural question then arises: is it possible to reconstruct the modified mass distribution from the observed perturbations of the vibration spectrum? Our purpose in this section of the paper is to discuss a first order approach to this problem from a mathematical point of view. Some theoretical studies yielding more or less global uniqueness results for mass/elasticity distributions versus frequency spectra obtained with various boundary conditions have been presented by McLaughlin [10], [11].

Let us consider a second order linear system in a Hilbert space X:

$$\ddot{y} + Ay = 0, \quad y \in \mathcal{D}(A) \underset{\text{(densely)}}{\subset} X,$$

where A is an unbounded self-adjoint operator on X with eigenvalues λ_k, $k = 1, 2, 3, \ldots$ of finite multiplicity and, without loss of generality, nondecreasing, with

$$\lim_{k \to \infty} \lambda_k = \infty,$$

the corresponding normalized eigenvectors ϕ_k, $k = 1,2,3,\ldots$, being selected so as to form an orthonormal basis for X. We let

$$\omega_k = \sqrt{\lambda_k}, \quad \lambda_k \geq 0, \quad \omega_k = i\sqrt{-\lambda_k}, \quad \lambda_k < 0.$$

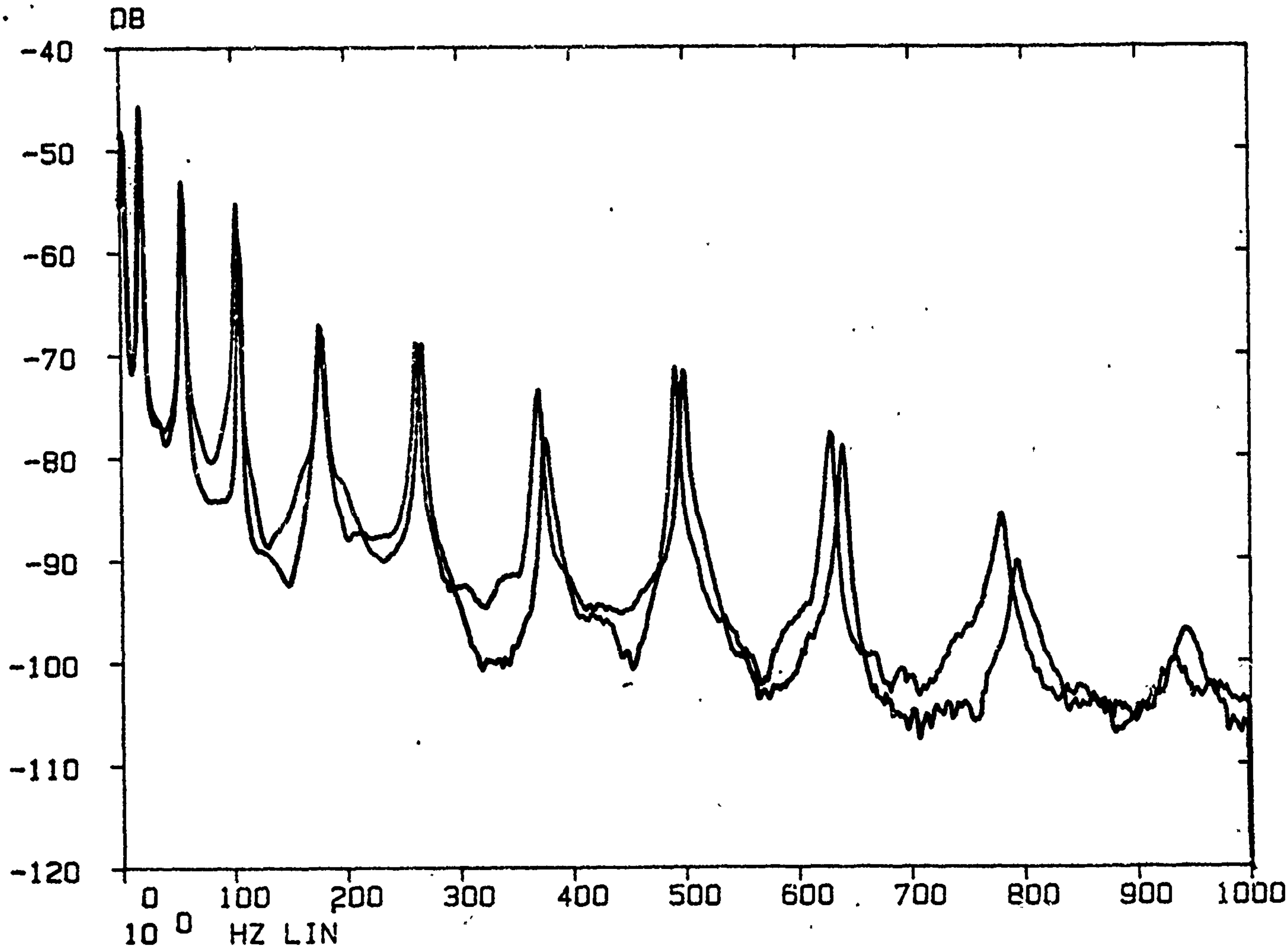
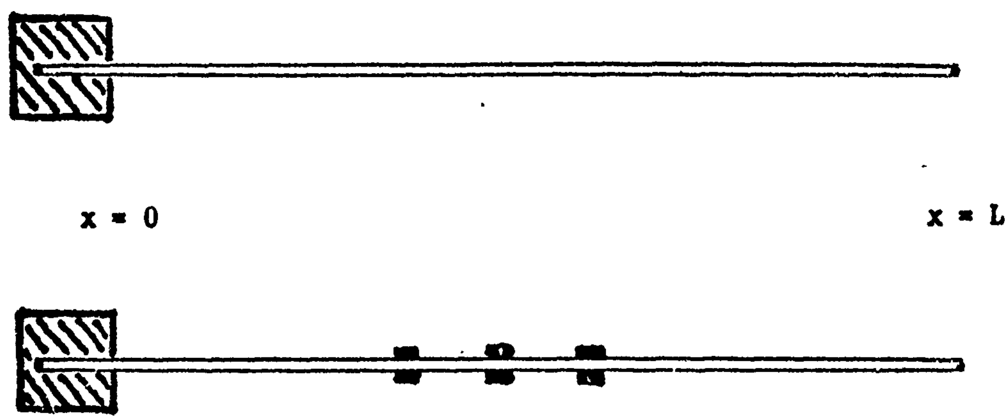


Figure 14

(a) Clamped / Free Beam in Original Configuration.



(b) Clamped / Free Beam with Added Permanent-Magnet Masses.

Figure 15

As is well known, the above second order equation is associated with the first order system

$$\frac{d}{dt}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} \qquad (x = \dot{y})$$

and the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$$

generates a strongly continuous group of bounded operators on Y x X, where Y is the closure of $\mathcal{D}(A) \subset X$ with respect to $\| \ \|_Y$, the norm associated with the inner product initially defined by

$$(y,\hat{y})_Y = (y,(\rho I + A)\hat{y})_X, \quad y, \hat{y} \in \mathcal{D}(A),$$

being selected to the $\rho I + A$ is strictly positive.

A scalar observation

$$z(t) = (y(t), \eta)_Y + (x(t), \xi)_X, \ \eta \in Y, \ \xi \in X,$$

$$\xi = \sum_{k=1}^{\infty} \xi_k \phi_k, \quad \eta = \sum_{k=1}^{\infty} \eta_k \phi_k,$$

$$\sum_{k=1}^{\infty} |\xi_k|^2 < \infty, \quad \sum_{k=1}^{\infty} (\rho + \lambda_k) |\eta_k|^2 < \infty,$$

is a continuous function of t which may be expressed, assuming it to be real valued, in the form

$$z(t) = \sum_{k=1}^{\infty} z_k e^{i\omega_k t} + \sum_{k=1}^{\infty} \bar{z}_k e^{-i\omega_k t}.$$

If the operator A is perturbed to the operator $A + \delta A$, having the same properties, but with eigenvalues $\lambda_k + \delta\lambda_k$, "frequencies" $\omega_k + \delta\omega_k$, and (non-normalized) eigenvectors $\phi_k + \delta\phi_k$, $\delta\phi_k \in [\phi_k]^\perp$, and if the same observation process is used, there will result the modified scalar observation

$$\zeta(t) = \sum_{k=1}^{\infty} \zeta_k e^{i(\omega_k + \delta\omega_k)t} + \sum_{k=1}^{\infty} \bar{\zeta}_k e^{-i(\omega_k + \delta\omega_k)t} .$$

Through the use of discrete Fourier techniques the frequencies $\omega_k + \delta\omega_k$ (real exponents if $\lambda_k < 0$) can be estimated, though with some additional difficulty if there actually are $\lambda_k < 0$. Then the identification question which we wish to address is this: assuming the λ_k (hence the ω_k) and the ϕ_k known (i.e., the operated A is assumed to be known completely), and assuming the $\omega_k + \delta\omega_k$ to be "adequately estimated", which means that the $\delta\lambda_k$ have been estimated, can we determine, at least in first order approximation, the operator perturbation δA?

Before going on, we should remark at this point that <u>any</u> identification technique for δA based only on <u>observations</u> $\zeta(t)$ must, in effect, address the same problem, for the $\zeta_k, \bar{\zeta}_k$ are, except for rates of decay, etc., arbitrary sequences if the solution of the system is unrestricted and, therefore, the only real information available consists of the $\delta\omega_k$, equivalently the $\delta\lambda_k$, albeit perhaps in an indirect form.

The first order analysis (see [7], [13] for more details) proceeds as follows. Accepting the equations

$$(-(\lambda_k + \delta\lambda_k)I + (A + \delta A))(\phi_k + \delta\phi_k) = 0, \; k = 1,2,3,\ldots,$$

we take the inner product of each left hand side with the corresponding ϕ_k to obtain, taking into account $\delta\phi_k \in [\phi_k]^{\perp}$,

$$(\phi_k, \delta A \phi_k)_X = \delta\lambda_k, \; k = 1,2,3,\ldots$$

In some cases the equation is originally given in the form

$$My + Ay = 0$$

with M bounded, self-adjoint, and strictly positive. To study variations $M \rightarrow M + \delta M$, A remaining fixed, one now takes the ϕ_k to be the solutions of the modified eigenvalue problem $(-\lambda M + A)\phi_k = 0$. The corresponding equation is then

$$(\emptyset_k , M\emptyset_k)_X \, \delta\lambda_k + \lambda_k (\emptyset_k , \delta M\emptyset_k)_X = 0, \; k = 1,2,3,\ldots$$

Assuming the $\emptyset_k$ normalized relative to M, the above becomes

$$(\emptyset_k , M\emptyset_k) = - \delta\lambda_k / \lambda_k = - \log\lambda \Big|_{\lambda = \lambda_k} ,$$

which we can use in the same way to infer properties of the perturbation δM. This second problem is usually simpler than the first and will be used for most of the examples which we cite here.

Consider the more specific case wherein $\ddot{y} + Ay = 0$ is the abstract version of the partial differentail equation

$$m(x)y_{tt} + \mathcal{L}y = 0, \; t \geqslant 0, \; 0 \leq x \leq L,$$

where m(x) is the mass per unit length of the medium occupying the spatial interval [0,L] and $\mathcal{L}$, the elasticity operator, is paired with boundary conditions making it a self-adjoint operator. If the normalized eigenfunctions are $\emptyset_k(x)$, $k = 1,2,3,\ldots$, i.e.,

$$\int_0^L m(x)\emptyset_k(x)\emptyset_j(x)dx = \delta_{kj} ,$$

then the problem of determining to first order a small mass variation m(x) leads to the infinite set of equations

$$\int_0^L m(x)\emptyset_k(x)^2 dx = -\delta\lambda_k / \lambda_k , \; k = 1,2,3,\ldots,$$

constituting a <u>moment problem</u> on $0 \leq x \leq L$. We will assume that

$$\lim_{k \to \infty} \delta\lambda_k / \lambda_k = \Lambda$$

exists. Then, since

$$\int_0^L m(x)(\emptyset_k(x))^2 dx = 1, \; k = 1,2,3,\ldots,$$

replacing $\delta m(x)$ by $\delta m(x) - \Lambda m(x)$ we arrive at a similar problem with Λ replaced by zero. Thus, without loss of generality, we may assume

$$\lim_{k\to\infty} \delta\lambda_k/\lambda_k = 0.$$

The rate at which $\delta\lambda_k/\lambda_k$ tends to zero will ordinarily determine our choice of function space for $m(x)$ to lie in. It should be noted that if we think of $\delta m(x)$ as a mass rotating about the x-axis at a distance $\phi_k(x)$, then the expression

$$\int_0^L \delta m(x)(\phi_k(x))^2\,dx$$

is, in fact, the moment of inertia of $\delta m(x)$ about the x-axis. Thus the moment problem in fact consists of finding $\delta m(x)$ with moments of inertia $-\delta\lambda_k/2\lambda_k$, $k = 1,2,3,\ldots$ (assuming no $\lambda_k = 0$, of course).

In this paper we will confine attention to the two questions of existence and uniqueness. For the existence question we suppose the $\delta\lambda_k$ arbitrary, subject to $\delta\lambda_k/\lambda_k$ having an appropriate rate of decay as $k\to\infty$, and ask if a solution $m(x)$ exists in a corresponding function space. The uniqueness question is posed in an equally self-evident way. It turns out that uniqueness essentially never obtains without further restriction on the form of $\delta m(x)$. The resolution of the existence question varies from one example to another. It is, in reality, the question of the linear independence of the functions $\phi_k^2(x)$ in the appropriate space, which we usually take to be $L^2[0,L]$. If the functions $\phi_k^2(x)$, $k = 1, 2,3,\ldots$, are strongly independent, i.e., there exist biorthogonal functions $\psi_k(x)$, $k = 1,2,3,\ldots$, also in $L^2[0,L]$, such that

$$\int_0^L \psi_k(x)\,\phi_j^2(x)\,dx = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}$$

then we have a formal solution

$$m(x) = -\sum_{k=1}^{\infty}\left(\delta\lambda_k/\lambda_k\right)\psi_k(x),$$

whose convergence, depending on the properties of the $\psi_k(x)$, can be investigated subsequently.

There is nothing about the problem in the abstract which can guarantee that the linear independence referred to above obtains in general. If we take $m(x) = 1$, $L = 2\pi$, and

$$\mathcal{L}y = -y''$$

with periodic boundary conditions

$$y(0) = y(2\pi),$$

then the eigenvalues are $\lambda_o = 0$, $\lambda_{2k} = \lambda_{2k-1} = k^2$, $k = 1,2,3,\ldots;$ all but λ_o having multiplicity 2 and the corresponding eigenfunctions are

$$\phi_{2k}(x) = \frac{\cos(kx)}{\sqrt{\pi}}, \quad -\phi_{2k-1}(x) = \frac{\sin(kx)}{\sqrt{\pi}}, \quad k = 1,2,3,\ldots,$$

while the simple eigenvalue 0 is associated with the eigenfunction $\phi_o(x) = \frac{1}{\sqrt{2\pi}}$.

The squared eigenfunctions are then

$$1/2\pi, \quad (1/\pi)\cos^2(kx), \quad (1/\pi)\sin^2(kx), \quad k = 1,2,3,\ldots,$$

and, since $(1/\pi)\cos^2(kx) = (1/\pi) - (1/\pi)\sin^2(kx)$, this set of functions is certainly not linearly independent in any sense. In this case a necessary condition for solvability of the moment problem is clearly

$$\delta\lambda_o = 0, \quad \delta\lambda_{2k} = -\delta\lambda_{2k-1}$$

and a general existence result for solutions of the moment problem cannot be obtained for arbitrary frequency perturbations via a first order analysis.

What may fairly be regarded as a "prototype" case arises again for $m(x) \equiv 1$, $L = \pi$, and

$$y = -y''$$

but with separated boundary conditions

$$a_o y(0) + b_o y'(0) = 0, \quad a_1 y(\pi) + b_1 y'(\pi) = 0,$$

$$a_o^2 + b_o^2 \neq 0, \quad a_1^2 + b_1^2 \neq 0.$$

In all of these cases the eigenvalues are non-negative and simple, the zero eigenvalue $\lambda_o = 0$ occurring only in the case $a_o = a_1 = 0$. There are four principal cases to consider:

<u>Case 1</u>: $a_o = a_1 = 0$. Here $\lambda_o = 0$, $\lambda_k = k^2$, $k = 1,2,3,\ldots$, and

$$\phi_o(x) = 1/\sqrt{\pi}\,,\quad \phi_k(x) = (1/\sqrt{2\pi})\cos(kx).$$

<u>Case 2</u>: $b_o \neq 0$, $b_1 \neq 0$, $a_o^2 + a_1^2 \neq 0$. Here the eigenvalues, all positive, have the form

$$\lambda_k = k^2 + \mathcal{O}(1),\ k = 0,1,2,\ldots,$$

and with

$$\omega_k = \sqrt{\lambda_k} = k + \mathcal{O}(1/k)$$

the eigenfunctions take the form

$$\phi_k(x) = \nu_k \cos(\omega_k x + \delta_k),$$

where

$$\nu_k = (1/\sqrt{2\pi}) + \mathcal{O}(1/k),\quad \delta_k = \mathcal{O}(1/k).$$

<u>Case 3</u>: $b_o = 0$, $b_1 \neq 0$ (the "mirror image" case $b_o \neq 0$, $b_1 = 0$ will not be treated separately). Here the λ_k are again all positive and, asymptotically,

$$\lambda_k = (k-\tfrac{1}{2})^2 + \mathcal{O}(1),$$

while the eigenfunctions and frequencies are given by

$$\phi_k(x) = \nu_k \sin(\omega_k x),\ k = 1,2,3,\ldots,$$

$$\omega_k = (k-\tfrac{1}{2}) + \mathcal{O}(1/k),\ \nu_k = (1/\sqrt{2\pi}) + \mathcal{O}(1/k),\ k = 1,2,3,\ldots\ .$$

<u>Case 4</u>: $b_o = b_1 = 0$. Here $\lambda_k = k^2$, $k = 1,2,3,\ldots$, and

$$\phi_k(x) = (1/\sqrt{2\pi})\sin(kx),\ k = 1,2,3,\ldots\ .$$

As noted earlier, in all cases we may assume the variation $\delta m(x)$ replaced by $m_o + \delta m(x)$, where m_o is a constant and

$$\int_0^{\pi} \delta m(x)dx = 0.$$

Since $(1/2\pi)\cos^2(kx) = (1/2\pi) - (1/2\pi)\sin^2(kx)$ and $\delta m(x)$ has been assumed orthogonal to constants, Cases 1 and 4 may be treated together, without loss of generality as Case 1. Here

$$(1/2\pi)\cos^2(kx) = (1/4\pi)(1 + \cos(2kx)),$$

and again making use of the above orthogonality property, the moment problem reduces to

$$\int_0^{\pi} \delta m(x)\cos(2kx)dx = -4\pi\,\delta\lambda_k/k^2 \;,\; k = 1,2,3,\ldots\;,$$

the case corresponding to $\lambda_o = 0$ already having been taken care of, assuming $\delta\lambda_o = 0$. The functions $\cos(2kx)$ are, of course, orthogonal on $[0,\pi]$ and, hence, independent. Adjoining 1, sin2kx, $k = 1,2,3,\ldots$, and normalizing we have an orthonormal basis for $L^2[0,\pi]$. The unique solution of the moment problem orthogonal to constants and even in the sense

$$\delta m(\pi - x) = \delta m(x)$$

is, formally,

$$m(x) = -2\pi \sum_{k=1}^{\infty} (\delta\lambda_k/k^2)\cos(2kx)\;,$$

and is convergent in $L^2[0,\pi]$ as long as $\delta\lambda_k = \mathcal{O}(k^{(3/2)-\varepsilon})$.

If the even-ness requirement is removed, the solution is highly non-unique, since one can add to $\delta m(x)$ any function $\hat{m}(x)$ in $L^2[0,\pi]$ which is odd in the sense

$$\hat{m}(\pi - x) = -\hat{m}(x).$$

Case 3 is mathematically nontrivial but, fortunately, the work has already been done for us. It is known from the work of Paley and Wiener [12], Levinson [9], Schwartz [15], and many others, that the functions

$$\sin(2\omega_k x),\ \cos(2\omega_k x),\ \omega_k = (k-\tfrac{1}{2}) + \mathcal{O}(1/k),\ k = 1,2,3,\ldots$$

form a Riesz basis for $L^2[0,\pi]$, provided the ω_k are distinct, as in the case here. Among other things this implies the existence of a unique, biorthogonal, dual basis $p_k(x)$, $q_k(x)$, with

$$\int_0^\pi \sin(2\omega_k x)p_j(x)dx = \int_0^\pi \cos(2\omega_k x)q_j(x)dx = \delta_{kj},$$

$$\int_0^\pi \sin(2\omega_k x)q_j(x)dx = \int_0^\pi \cos(2\omega_k x)p_j(x)dx = 0.$$

Each function f in $L^2[0,\pi]$ has unique expansions

$$f(x) = \sum_{k=1}^{\infty}(c_k \sin(2\omega_k x) + d_k \cos(2\omega_k x)) = \sum_{k=1}^{\infty}(\hat{c}_k p_k(x) + \hat{d}_k q_k(x)),$$

convergent in $L^2[0,\pi]$ with

$$c_k = \int_0^\pi p_k(x)f(x)dx, \quad d_k = \int_0^\pi q_k(x)f(x)dx,$$

$$\hat{c}_k = \int_0^\pi \sin(2\omega_k x)f(x)dx, \quad \hat{d}_k = \int_0^\pi \cos(2\omega_k x)f(x)dx$$

and, for certain positive numbers R, r

$$r\|f\|^2_{L^2[0,\pi]} \le \sum_{k=1}^{\infty}(|c_k|^2 + |d_k|^2) \le R\|f\|^2_{L^2[0,\pi]},$$

$$(1/R)\|f\|^2_{L^2[0,\pi]} \le \sum_{k=1}^{\infty}(|c_k|^2 + |d_k|^2) \le (1/r)\|f\|^2_{L^2[0,\pi]}.$$

The moment problem

$$\int_0^\pi \delta m(x)\phi_k(x)^2 dx = -\delta\lambda_k/\lambda_k,\ k = 1,2,3,\ldots$$

in this case becomes

$$\nu_k^2 \int_0^\pi \delta m(x)\sin^2(\omega_k x)dx = -\delta\lambda_k/(k^2 + \mathcal{O}(1)),\ k = 1,2,3,\ldots,$$

and, since $\delta m(x)$ is assumed orthogonal to constants, this is the same as

$$\int_0^\pi \delta m(x)\cos(2\omega_k x)dx = 2\delta\lambda_k/(\nu_k^2(k^2 + \mathcal{O}(1)))$$

The unique solution lying in the space spanned by the functions q_j, $j = 1, 2, 3, \ldots$ is

$$m(x) = 2 \sum_{j=1}^{\infty} (\delta\lambda_j / (\nu_j^2 (j^2 + \mathcal{O}(1)))) q_j(x)$$

convergent just in case the sequence of coefficients is square summable. Again this solution is non-unique; one may add any convergent linear combination of the $p_j(x)$ and still have a solution of the moment problem. Here the displayed solution cannot be characterized as odd or even relative to the center point $\pi/2$ of the x-interval because the functions $\cos(2\omega_k x)$, $\sin(2\omega_k x)$ have no such properties, except in some asymptotic sense as $k \to \infty$.

Finally, Case 3 is the most complex because here

$$\phi_k^2(x) = \nu_k^2 \cos^2(\omega_k x + \delta_k) = \nu_k^2((1 + \cos(2\omega_k x + 2\delta_k))/2).$$

The constant term can be discarded for the same reasons as before and, since

$$\cos(2\omega_k x + 2\delta_k) = \cos(2\delta_k)\cos(2\omega_k x) - \sin(2\delta_k)\sin(2\omega_k x),$$

the relevant moment problem becomes

$$\alpha_k \int_0^{\pi} \delta m(x)\cos(2\omega_k x)dx + \beta_k \int_0^{\pi} \delta m(x)\sin(2\omega_k x)dx$$
$$= -(2\delta\lambda_k)/(\nu_k^2((k-\tfrac{1}{2})^2 + \mathcal{O}(1))),$$

where, as $k \to \infty$,

$$\alpha_k = \cos(2\delta_k) = 1 + \mathcal{O}(1/k^2),$$

$$\beta_k = -\sin(2\delta_k) = \mathcal{O}(1/k).$$

Perhaps the most natural solution here is

$$\delta m(x) = -2 \sum_{j=1}^{\infty} (\delta\lambda_j / (\nu_j^2((j-\tfrac{1}{2})^2 + \mathcal{O}(1)))) [\alpha_j q_j(x) + \beta_j p_j(x)]$$

with convergent linear combinations of the functions $\beta_j q_j(x) - \alpha_j p_j(x)$

constituting the space of functions which may be arbitrarily added to $\delta m(x)$, as displayed, without changing its property of being a solution of the moment problem.

If we maintain $m(x) = 1$ and suppose that the equation $m(x)y_{tt} + \mathcal{L}y = 0$ is, in reality,

$$y_{tt} - (p(x)y_x)_x = 0,$$

then perturbations $p(x) = 1 + \delta p(x)$ about the nominal $p_0(x) \equiv 1$ lead, as noted previously, to the equations

$$\int_0^{\pi} \phi_k(x)(\delta p(x)\phi_k'(x))'dx = \int_0^{\pi} \delta p(x)(\phi_k'(x))^2 dx = \delta\lambda_k, \; k = 1,2,3,\ldots,$$

which may be analyzed in much the same way as the problem for $\delta m(x)$ treated in detail above with comparable solutions and non-uniqueness aspects as were developed there.

To conclude this section on identification from frequency data, let us consider the case of the Euler-Bernoulli beam, which corresponds to

$$m(x)y_{tt} + \mathcal{L}y = 0$$

with

$$\mathcal{L}y = (EI(x)y_{xx})_{xx}$$

and appropriate boundary conditions, for example, the cantilever beam (the clamped/free case) corresponds to

$$y(0,t) = y_x(0,t) = 0,$$

$$y_{xx}(L,t) = (EI(x)y_{xx})_x \Big|_{x=L} = 0.$$

If we begin with $m_0(x) \equiv 1$, $EI(x) \equiv 1$, $L = \pi$, and consider $m(x) = 1 + \delta m(x)$ for small $\delta m(x)$ we are again led to the moment problem

$$\int_0^{\pi} \delta m(x)(\phi_k(x))^2 dx = -\delta\lambda_k/\lambda_k .$$

As we have seen earlier, if in this case we set $\omega_k = \sqrt{\lambda_k}$, the ω_k are the positive roots of

$$1 + \cos(\omega^{\frac{1}{2}}\pi)\cosh(\omega^{\frac{1}{2}}\pi) = 0$$

or

$$\cos(\omega^{\frac{1}{2}}\pi) = -(\cosh(\omega^{\frac{1}{2}}\pi))^{-1}$$

leading to

$$\omega_k^{\frac{1}{2}} = (k-\tfrac{1}{2}) + \mathcal{O}(e^{-k\pi}) .$$

Then from the equation

$$c(-\sin(\omega^{\frac{1}{2}}\pi) - \sinh(\omega^{\frac{1}{2}}\pi)) + d(-\cos(\omega^{\frac{1}{2}}\pi) - \cosh(\omega^{\frac{1}{2}}\pi)) = 0$$

it is clear that for $\omega = \omega_k$ we must have

$$c = c_k = \nu_k + \gamma_k e^{-\omega_k^{\frac{1}{2}}\pi}$$

$$d = d_k = -\nu_k + \delta_k e^{-\omega_k^{\frac{1}{2}}\pi}$$

where, if the eigenfunctions $\phi_k(x)$ are normalized,

$$\lim_{k\to\infty} \nu_k = \nu$$

for some positive constant ν and the γ_k, δ_k are uniformly bounded. Thus

$$\phi_k(x) = (\nu_k + \gamma_k e^{-\omega_k^{\frac{1}{2}}\pi})(\sin(\omega_k^{\frac{1}{2}}x) + e^{-\omega_k^{\frac{1}{2}}x})$$
$$+(-\nu_k + \delta_k e^{-\omega_k^{\frac{1}{2}}\pi})(\cos(\omega_k^{\frac{1}{2}}x) - e^{-\omega_k^{\frac{1}{2}}x}) + (\gamma_k - \delta_k)e^{-\omega_k^{\frac{1}{2}}(\pi-x)} .$$

From this it may be seen that $\phi_k(x)$ is a linear combination, with bounded coefficients, of the functions

$$\sin(\omega_k^{\frac{1}{2}}x),\ \cos(\omega_k^{\frac{1}{2}}x),\ e^{-\omega_k^{\frac{1}{2}}x},\ e^{-\omega_k^{\frac{1}{2}}(\pi-x)} .$$

In this case we know from the theory of the operator $\mathcal{L}$ itself that the $\phi_k(x)$ are orthonormal in $L^2[0,\pi]$, and hence independent, but no theorems are readily available concerning the independence properties of the functions $(\phi_k(x))^2$, which constitute the defining kernels for the moment problem equivalent to the first order frequency-based identification problem for $\delta m(x)$. Interest in this problem will thus lead to further study of the independence properties of complex exponentials and related functions (see [14], [16]).

The results noted above for the vibrating string all have the characteristic that, in terms of density of the exponents of the exponential functions involved, only half as many equations are available, based on observed frequency perturbations, as would be required to determine $\delta m(x)$, or $\delta p(x)$, completely. If a boundary control force can be implemented, consistent with the boundary condition

$$y_x(\pi,t) = u(t),$$

for example, we may obtain the complementary set of equations by determining u via boundary feedback. For example, with

$$u(t) = -\gamma y_t(\pi,t),$$

another set of perturbed spectra, and related moment equations, can be obtained, corresponding to the boundary condition

$$y_x(\pi,t) + y_t(\pi,t) = 0$$

and these, along with those obtained for $u(t) = 0$, can be shown to be sufficient to determine $\delta m(x)$, or $\delta p(x)$, completely. This is in agreement with the more global, but theoretical, results of Borg [1], [8].

It is likely that in most applications the identification problem will be less complicated than the one we have treated here. In many cases we may expect that it will be known in advance, for the mass perturbations $\delta m(x)$, for example, that

$$m(x) = \mu_1 m_1(x) + \mu_2 m_2(x) + \ldots + \mu_r m_r(x),$$

where $m_1(x), m_2(x), \ldots, m_r(x)$ are functions known in advance. This would be the case, for example, if the individual terms represented fuel or payload of unknown mass but stored in known configurations on the elastic body in question. In such cases the moment equations in fact overdetermine the coefficients μ_k to be identified; some sort of least squares solution has to be implemented, which leads to an unambiguous result provided that no nontrivial linear combination of the $m_k(x)$ is orthogonal to all of the squared eigenfunctions $(\phi_k(x))^2$.

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